HIGHER-ORDER LOGIC AS METAPHYSICS

JUHANI YLI-VAKKURI
Australian Catholic University

Based on joint work with Zachary Goodsell (USC)

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INTRODUCTION

- I want to illustrate how metaphysics is done nowadays within analytic philosophy.
- Here's how I'll do it:
- I am going to list some views that are widely held (sometimes, I maintain, tacitly) by contemporary metaphysicians. I'll call them "metaphysical orthodoxy".
- Then I am going to illustrate how contemporary metaphysicians might go about answering the question "Why should I believe that?" about each of the metaphysical orthodoxies.
- I don't mean to suggest that the answers I'm going to give are ones that you would hear from many metaphysicians.
- In fact, some of these answers are based on original research that isn't in print yet.
- (This talk is based on a paper that I am coauthoring with Zachary Goodsell.)
- But the answers I am going to give are examples of a kind of methodology that is very common in contemporary metaphysics.
- Namely, the methodology of *higher-order logic as metaphysics*.

METAPHYSICAL ORTHODOXY

- So let's begin by asking: What do metaphysicians believe?
- I don't know how common any of the views that I am going to list actually are among metaphysicians.
- For all I know, it may be that none of them is a view that's accepted by more than 50% of professional philosophers who specialize in metaphysics.
- But I think that people working in the field will recognize each of them (with one exception that's a little complicated) as a view that's treated as having a "default" status on the topic…
- … at least in the sense that you know that if you want to argue for a *surprising* conclusion regarding the topic, you should argue against that view rather than for it…
- … and each of these views (again with one exception) is one that we generally think it's OK to treat as an assumption in your work. Referees typically won't ask you to defend them.
- Here they are, numbered 1 to 11, in no particular order:

1. THERE IS A UNIQUE STRONGEST NECESSITY

- Kripke calls it "necessity in the highest degree" and "metaphysical necessity".
- I'll call it "$\Box$" and "metaphysical necessity".
- What does it mean to say that there is unique strongest necessity?
- Here's what I'll mean by it:
- First, necessities are properties of propositions that are (i) possessed by the canonical tautology and (ii) closed under *modus ponens:*
  - X is a necessity $\coloneqq$ (i) $X(\top)$ & (ii) $\forall p \forall q(X(p \rightarrow q) \rightarrow (Xp \rightarrow Xq))$
- Second,
- X is a strongest necessity $\coloneqq$ X is a necessity & $\forall Y \forall p(Y$ is a necessity $\rightarrow (Xp \rightarrow Yp))$

  & necessarily so (for every necessity).

2. THE LOGIC OF $\Box$ IS **S5**

- **S5** is the logic in which all iterated modalities collapse.
- **S5** is the closure of the following axioms under *modus ponens* and necessitation ($A/\Box A$):
  - All tautologies
  - $\Box(A \to B) \to (\Box A \to \Box B)$ (axiom K)
  - $\Box A \to A$ (axiom T)
  - $\sim\Box A \to \Box\sim\Box A$ (axiom 5) [Equivalently: $\Diamond A \to \Diamond\Box A$.]
- **S5**-theorems include:
  - $\Box A \leftrightarrow \Box\Box A$
  - $\Box A \leftrightarrow \Diamond\Box A$
  - $\Diamond A \leftrightarrow \Box\Diamond A$
  - $\Diamond A \leftrightarrow \Diamond\Diamond A$ [‘$\Diamond$’ (‘metaphysically possibly’) is short for ‘$\sim\Box\sim$’.]

3. THE NECESSITY OF IDENTITY (**NI**)

- $\forall x \forall y(x = y \rightarrow \Box x = y)$

Identical things are metaphysically necessarily identical.

4. THE NECESSITY OF DISTINCTNESS (**ND**)

- $\forall x \forall y(x \neq y \rightarrow \Box x \neq y)$
- Distinct things are metaphysically necessarily distinct.

5. □ IS TRUTH IN ALL POSSIBLE WORLDS

- What does this mean?
- Think of this as a generalization of the first two propositions of the *Tractatus Logico-Philosophicus*:
  - "The [actual] world is everything that is the case;"
    - "The world is the totality of facts, not things."
- Contemporary metaphysicians tend to think that there is not just the actual totality of facts but many possible totalities of facts.
- The possible worlds are all the collections of states of affairs/propositions that could have been all the facts
- … and to be metaphysically necessary is to be true in (or simply *in*) one of those collections.

6. THERE IS AN ACTUALITY OPERATOR (@)

- Think of '@*A*' ('actually *A*') as saying that, in the actual world, *A*.
- Or, in Wittgensteinian terms: *A* is included in the actual totality of facts.
- The key axiom of the logic of @ is the axiom of Rigidity:
  - $A \rightarrow \Box @A$
- Other axioms:
  - $@A \rightarrow A$
  - $\Box(\sim @A \leftrightarrow @\sim A)$
  - $\Box(@(A \rightarrow B) \leftrightarrow (@A \rightarrow @B))$

7. CLASS COMPREHENSION

- *For every property P there is a unique class C of things that have P,*
  - *such that x is a member of C iff P(x)*
- Class Comprehension is the exceptional member of my list.
- Class Comprehension is widely believed to be inconsistent (Russell's paradox).
- But metaphysicians often use the language of 'classes', 'collections', 'sets', 'pluralities', 'totalities' (etc.) in a way that seems to commit them to Class Comprehension in *some* sense of 'class'.
- For example: I've been talking about the totality of all facts and all collections that could have been totalities of facts.

8. EXTENSIONALITY

- *Classes that have the same members are the same.*

- Most metaphysicians are probably used to thinking of Extensionality as an axiom.
- But you can still reasonably ask why it's true.
- There might be a more basic axiom from which it can be derived.

9. THE NECESSITY OF CLASS MEMBERSHIP

- x is a member of A $\rightarrow \Box$(x is a member of A)

10. THE NECESSITY OF CLASS EXCLUSION

- x is not a member of A $\rightarrow \Box$(x is not a member of A)

11. THE NECESSITY OF MATHEMATICS

- If *A* is a statement of pure mathematics, then:
  - $\Box A \vee \Box \sim A$

EARLIER APPROACHES TO METAPHYSICAL ORTHODOXY

- In the late 20th century metaphysicians tended to use a variety of different methods when they asked why (or whether) orthodoxies like these are true.
- For example: the methods of modal logic—esp. possible worlds semantics—and set theory and other first-order mathematical theories.
- The results of these investigations were very inconclusive.
- For example, possible worlds semantics can only tell you that a set of modal axioms is consistent. It can't give you a reason to accept the axioms.
- And custom-made modal axioms look *ad hoc* and are hard to motivate.
- The same is true of custom-made axioms about classes/collections/pluralities/etc.
- Even more so for mixed axioms like $\forall x(x \in A \rightarrow \Box x \in A)$.

CURRENT APPROACHES

- In the early 21st century there is a tendency to instead look to higher-order logic as the foundation of metaphysics.
- (There was a similar tendency in the early 20th century in the work of Russell and Church.)

HIGHER-ORDER METAPHYSICS

- **Some examples of work along these lines**

- T. Williamson, "Everything", *Philosophical Perspectives*, 2003
- T. Williamson, *Modal Logic as Metaphysics*. Oxford University Press, 2013
- C. Dorr, "To be F is to be G", *Philosophical Perspectives*, 2016
- A. Bacon, J. Hawthorne, & G. Uzquiano, "Higher-order free logic and the Prior-Kaplan paradox", in J. Yli-Vakkuri & M. McCullagh, eds., *Williamson on Modality*. Routledge, 2017
- P. Fritz & J. Goodman, "Higher-Order Contingentism, Part 1", *Journal of Philosophical Logic*, 2016
- J. Goodman, "Reality is not structured", *Analysis*, 2017
- A. Bacon, "The Broadest Necessity", *Journal of Philosophical Logic*, 2018
- A. Bacon, *A Philosophical Introduction to Higher-Order Logic.* Oxford University Press, forthcoming
- P. Fritz, "Ground and Grain", In preparation.
- C. Dorr, J. Hawthorne, & J. Yli-Vakkuri, *Living on the Edge: Puzzles of Modal Variation*. Oxford University Press, forthcoming (recently renamed *The Bounds of Possibility*)
- Z. Goodsell and J. Yli-Vakkuri, "Higher-Order Logic as Metaphysics", in preparation

HIGHER-ORDER LOGIC

- Let's now look at how we might try to justify metaphysical orthodoxy using the higher-order approach.
- We are interested in strengthenings of classical HOL.
- (A. Church, "A formulation of the simple theory of types", *JSL*, 1940).
- That is (roughly) first order logic generalized to all types, so that we can quantify into the position of any type of expression, as in:
- $\forall X(Xa \;\&\; {\sim}Xa)$, $\forall p(p \;\&\; {\sim}p)$, $\Box\varphi \rightarrow \exists X(X\varphi)$
- And we can write an identity predicate $=_\sigma$ between any two expressions of the same type σ.

HIGHER-ORDER LOGIC

- In particular, I want to look at strengthenings of classical HOL that do *not* include the axiom of extensionality, which says that coextensional properties are the same…
- … but that *do* include the axiom of Functionality:
  - $\forall x(Fx =_\tau Gx) \rightarrow F =_\tau G$
    - ($F\colon \sigma \rightarrow \tau$, $x\colon \sigma$),
  - which says that functions that have the same value for every argument are the same
- (Functionality is from Church 1940.)

HIGHER-ORDER LOGIC

- The relevant strengthenings of classical HOL are also closed under the Rule of Equivalence:
  - If $\vdash A \leftrightarrow B$ then $\vdash A =_t B$
- That is: Propositions that are materially equivalent according to the logic are also identical according to it.
- (The Rule of Equivalence was proposed by Church in 1943.)

HFE

- In the weakest such logic, HFE, we can prove two orthodoxies:
  - There is a strongest necessity.
    - The Necessity of Identity.
- This was shown by Bacon (2018) (although using different definitions).
- But we can’t prove any of the other orthodoxies in HFE.

THE AXIOM OF CHOICE

- But it turns out that if we add just one axiom, to our classical higher-order logic, we can prove all of metaphysical orthodoxy.
- That is the axiom of choice (AC):
  - $\exists f^{(\sigma \to t) \to \sigma} \forall X^{\sigma \to t} (\exists y Xy \to X(fX))$
    - (σ is any type; t is the type of a formula.)
- AC says that there is a *choice function*, a function that takes every property that something has to a thing that has it.
- This a standard axiom used in higher-order formalizations of mathematics.
- (It's also in Church 1940, though not exactly in this form.)

HCE

- HCE is classical higher-order logic with AC, closed under the Rule of Equivalence.
- Let’s look at metaphysical orthodoxy in HCE, beginning with classes.

CLASSES

- The basic idea: A class is a special kind of property, a property that's in the strictest sense inseparable from its instances.
- A property *P* is a class iff, (i) whenever something *x* has *P*, the fact that *x* has *P* is a tautology, and whenever (ii) *x* doesn't have *P*, the proposition that *x* has *P* is a contradiction:
    - $\text{Class}(X) := \forall x[(Xx \rightarrow Xx = \top) \,\&\, ({\sim}Xx \rightarrow Xx = \bot)]$
- Now I'm going to give you a sketch of a proof of Class Comprehension.

CLASSES

- Let P be any property and x any object.
- Consider the following property of propositions:
  - $C_{P,x}$:= being a proposition p such that $p = \top$ if x has P and otherwise $p = \bot$.
    - [formally: $C_{P,x} := \lambda p \,.\, (Px \rightarrow p = \top) \,\&\, (Px \rightarrow p = \bot)$]
- By classical logic, exactly one thing has the property $C_{P,x}$: either $\top$ or $\bot$ has it.
- Since something has the property $C_{P,x}$, by AC, the choice function ***choice*** takes $C_{P,x}$ to something that has $C_{P,x}$:
  - $C_{P,x}(\mathit{choice}(C_{P,x}))$
- So,
  - $\mathit{choice}(C_{P,x}) = \top$ if x has P
  - $\mathit{choice}(C_{P,x}) = \bot$ otherwise
- Now consider the function that takes x to the value of the choice function for $C_{P,x}$:
  - $x \longmapsto \mathit{choice}(C_{P,x})$
  - [or: $\lambda x \,.\, \mathit{choice}(C_{P,x})$]
- It is a function that takes x to $\top$ if x has P and otherwise takes x to $\bot$. That's a class.

CLASSES

- I just *almost* proved Class Comprehension: I proved that every property is coextensional with a class.
- Class Comprehension says that every property is coextensional with a *unique* class.
- This we can also prove, since in HCE we can prove that functions that give the same values for every argument are identical.
- That takes care of **Class Comprehension**.
- That also takes care of **Extensionality**:
- Functions that give the same values for every argument are identical.
- Classes are functions that take their members to $\top$ and their non-members to $\bot$.
- So, same members, same classes (Extensionality).

THERE IS A STRONGEST NECESSITY

- Metaphysical necessity turns out to be the singleton of the tautology.
- Adopt the definition
  - $\Box := \{\top\}$,
- and you can prove in HCE that $\Box$ is the strongest necessity.
- (**NB:** You can also prove $\{\top\} =_t \lambda p(p =_t \top)$, and generally $\{x\} =_t \lambda y(y =_t x)$. The singleton of *x* is the property of being *x*.)

THE NECESSITY OF IDENTITY (NI)

- Like the previous result, this is something we can prove without AC.
- The basic idea of the proof of NI is already found in the work of Kripke (1971), probably following Quine, who in turn is probably following and Barcan Marcus (according to John Burgess).
- The idea: NI follows from the necessity of self-identity and Leibniz's law.

THE NECESSITY OF DISTINCTNESS (ND)

- But for ND we need the full power of HCE.
- Proof hint: The singleton of an object is a function that takes it to the tautology and everything else to the contradiction.

THE LOGIC OF ☐ IS **S5**

- S5 follows from ND.
- The idea: In HCE we can show that to be *not* necessary is to be *distinct* from the contradiction.
- The characteristic axiom of S5 says that whatever is not necessary is necessarily not necessary.

☐ IS TRUTH IN ALL POSSIBLE WORLDS

- This turns out to be very straightforward. Recall:
    - "The [actual] world is everything that is the case;"
        - "The world is the totality of facts, not things."
- The totality of facts is $\{p : p\}$, the class of all propositions p such that p. But there are also classes of propositions that could have been totalities of facts but aren't.
- Those are all the possible worlds.
- Being *true in* a possible world just amounts to being a member of it.

THERE IS AN ACTUALITY OPERATOR (@)

- This one's easy too:
- @ := $\{p : p\}$
- The actuality operator is the actual world, the class of all facts.
- You can derive all the axioms of @ in HCE using that def.
- Now let's look at classes and their modal properties.

THE NECESSITY OF CLASS MEMBERSHIP

- This is immediate from the definition of class.
- If *C* is a class and *x* is in *C*, then by definition
  - (x is in C) $=_t$ $\top$,
- and in HCE being identical to $\top$ is metaphysical necessity.

THE NECESSITY OF CLASS EXCLUSION

- This is also immediate from the definition of class.
- If C is a class and x is *not* in C, then by definition
    - (x is in C) $=_t \perp$,
- so
    - $\sim$(x is in C) $=_t \top$.
- And, again, in HCE being identical to $\top$ is metaphysical necessity.

THE NECESSITY OF MATHEMATICS

- Here's a more general fact:
- **The Necessity of Logic.** Whenever *A* is purely logical (contains no occurrences of non-logical constants), we can prove in HCE:
  - $\Box A \vee \Box {\sim} A$
- (Unfortunately I don't have any snappy proof sketch or hint for this one. The proof doesn't seem to lend itself to such a thing.)
- So, provided that we formalize mathematics using purely logical sentences of higher-order logic (there are standard ways of doing this) the necessity of mathematics follows.

REFLECTIONS ON THE FOREGOING

- The set-theoretic version of the axiom of choice is famous for its strength.
- We can get a lot of mathematics out of it …
- … and we need it to prove lots of truths that we take for granted in various areas of mathematics but that it would be awkward to treat as separate axioms.
- When we take the higher-order perspective, we notice that the higher-order axiom of choice may have a similar role in metaphysics.

HIGHER-ORDER LOGIC AS METAPHYSICS

- THANK YOU